Nonplanar modes cannot be considered here. Let us just note that they are known to exist. Thus modes corresponding to a circular orbit are found by elementary means. Hence $u = \cos\tau$, $w = \pm\sin\tau$, $\varkappa l = 2\pi, 4\pi$, etc., and the form of the string is a helical line having n curls for $\varkappa l = 2\pi n$. Such modes exist in pairs: as right-hand and left-hand spirals, which corresponds to two directions of body rotation around the center. Seeking the remaining modes is substantially more complex than determining the periodic motions in the Newtonian potential case, for example. This is seen at least from the fact that an unrealizable quadrature will replace the equations of the conic sections.

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PROPAGATION OF A SHOCK WAVE IN A CHANNEL WHEN SHOCK-COMPRESSED GAS INTERACTS WITH A NONHOMOGENEOUS MAGNETIC FIELD

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Unsteady flow of a conducting gas under shock wave conditions in channels of various magnetohydrodynamic devices was investigated in several recent papers (see for example [1, 2]). Most of them assume that the electrical current distribution in the gas behind the shock wave is one-dimensional, and that it is controlled by the conditions of current closure in the external electrical circuit that connects the electrodes at the channel walls.

However, in real channels there are always regions where the magnetic field is nonhomogeneous and where the channel walls are nonconducting. As a rule, these regions coincide with the end zones of the external magnetic field. Behind the shock wave passing through the end zones in the gas there are closed electrical currents whose intensity depends on the position of the shock front. These two-dimensional currents interact with the magnetic field and cause perturbations which catch up with the shock wave and change its velocity.

Terminal effects in steady magnetohydrodynamic flows have been investigated for a long time (see for example [3]) but their influence on the unteady gas flow has not yet been solved definitely. Among the papers devoted to this subject matter are two experimental studies [4, 5] which indicate that a substantial change occurs in the velocity of the plasma front in nonhomogeneous magnetic field, and that this effect is related to the emergence of closed-current zones in the plasma.

Our purpose here is to analyze theoretically some aspects of this problem. Its mathematical complexity is enormous (the corresponding magnetohydrodynamic equations are two-dimensional and nonstationary) and, with a view to obtaining more readily intelligible results, we assume that the parameter of magnetohydrodynamic interaction is less than unity, and that the induced magnetic fields have no effect on the plasma motion. It should be noted that these conditions frequently occur in magnetohydrodynamic systems, when the gas conductivity is of the order of 1 mho/cm and when the velocities and magnetic fields are not very high.

1. Let us consider plane or axisymmetric gas motion behind a shock wave propagating in a gas at rest in a channel $|x^\circ| < \infty$, $0 < y^\circ < h = \text{const}$, whose walls are nonconducting in the presence of an external steady field $\mathbf{B}^\circ$. The equations of magnetogasdynamics describing the flow under such conditions are as follows:

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\nu\rho v}{y} = 0 \qquad \left(N = \frac{\sigma_* B_*^2 h(\gamma-1)}{c^2\rho_\infty(\gamma+1)D_*}\right) \tag{1.1}$$

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} = N(\mathbf{j}\times\mathbf{b})_x \tag{1.2}$$

$$\rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} = N(\mathbf{j}\times\mathbf{b})_y \tag{1.3}$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \gamma p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\nu v}{y}\right) = N\frac{\gamma-1}{\sigma}\mathbf{j}^2 \tag{1.4}$$

$$\mathbf{j} = \sigma(-\nabla\varphi + \mathbf{v}\times\mathbf{b}), \qquad \operatorname{div}\mathbf{j} = 0 \tag{1.5}$$

$$\left.\begin{aligned} v_n &= \frac{2D}{\gamma+1}\left(1-\frac{s}{D^2}\right), \qquad v_t = 0 \\ \rho &= \left[1+\frac{2s}{(\gamma-1)D^2}\right]^{-1}, \qquad p = \frac{(\gamma-1)s}{\gamma(\gamma+1)} + \frac{(\gamma-1)Dv_n}{\gamma+1}\end{aligned}\right\} \left(s = \frac{a_\infty^2}{D_*^2},\ a_\infty^2 = \frac{\gamma p_\infty}{\rho_\infty}\right) \text{(on the shock wave)} \tag{1.6}$$

$$j_n = 0 \quad \text{on the shock wave} \tag{1.7}$$

$$v = 0, \quad j_y = 0 \quad \text{for } y = 0,\ y = 1 \tag{1.8}$$

All quantities in Eqs. (1.1) – (1.8) are dimensionless. The dimensional (with a superscript "°") and dimensionless (without superscript) variables are interrelated as follows:

$$\begin{gathered} x^\circ = hx, \quad y^\circ = hy, \quad t^\circ = \frac{ht}{D_*}, \quad D^\circ = D_*D \\ \mathbf{v}^\circ = D_*\mathbf{v}, \quad \rho^\circ = \frac{\gamma+1}{\gamma-1}\rho_\infty\rho, \quad p^\circ = \frac{\gamma+1}{\gamma-1}\rho_\infty D_*^2 p \\ \mathbf{B}^\circ = B_*\mathbf{b}, \quad \mathbf{j}^\circ = \frac{\sigma_*}{c}B_*D_*\mathbf{j}, \quad \varphi^\circ = \frac{h}{c}B_*D_*\varphi \end{gathered} \tag{1.9}$$

Here t is the time, ρ the density, p the pressure, D the velocity at which the snock wave propagates in space, v_n and j_n the projections of the vectors of absolute velocity of the gas and of the electrical current density onto the normal $\mathbf{n}$ to the shock wave ($\mathbf{n}$ is directed towards the medium at rest), u and v the projections of the absolute velocity vector $\mathbf{v}$ onto the x- and y-axes, φ the electrostatic potential (associated with electrical field defined in the fixed coordinate system), σ the electrical conductivity of the gas, c the light velocity in vacuum, and γ the ratio of specific heats;

$\nu = 0$ for plane flow and $\nu = 1$ for axisymmetric flow. The dimensional quantities D_*, $\mathbf{B}_*$ and σ_* represent, respectively, the shock velocity before the region in which the interaction with the magnetic field takes place, the characteristic external magnetic field, and the conductivity of the gas. The dimensional pressure and density of the gas at rest are denoted by p_∞ and ρ_∞.

The quantity N is the magnetohydrodynamic interaction parameter; the parameter s is equal to the reciprocal of the squared Mach number, calculated from the characteristic velocity of the shock wave and the velocity of sound a_∞ in the gas at rest.

Equations (1.1) – (1.8) were derived assuming that the medium is a perfect gas with constant specific heats, that the induced magnetic fields and Hall effect are fairly small, that the channel walls are impermeable to the gas, and that the electrical conductivity of the medium before the shock wave is nil.

Let us consider some characteristics of the motion under investigation.

First of all, it must be noted that external magnetic field $\mathbf{b}$ in real devices is concentrated in some zone L of finite length and is virtually equal to zero outside this zone (Fig. 1). Until the shock wave S enters the zone L $(t < 0)$ the gas flow behind the shock front remains homogeneous (since $\mathbf{j} = 0$ and $\varphi = 0$), and the shock wave velocity stays constant $(D = 1)$.

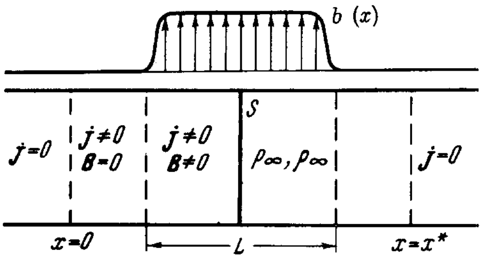


Fig. 1

The gas temperature behind intense shock waves rises to such an extent that the conductivity becomes significant. Hence, once the shock wave has entered the zone L $(t > 0)$, electrical currents $(\mathbf{j} \neq 0)$ begin to flow in the moving gas because of its interaction with the magnetic field. The distributions of $\mathbf{j}$ and φ under these conditions can be found from Eqs. (1.5) which are elliptical, so that in principle $\mathbf{j}$ and φ are "excited" instantaneously in the entire stream behind the shock wave. However, in practice these quantities reach significant values only to the right of some cross section $x = 0$ which is generally higher upstream than zone L, at a distance of the same order as the size of the channel (Fig. 1). To the left of this cross section quantities $\mathbf{j}$ and φ are small and may be neglected.

When solving Eqs. (1.5), only the region between the shock wave and the cross section $x = 0$ needs to be considered, and the following condition can be applied:

$$j_x = 0 \quad \text{at} \quad x = 0,\ 0 < y < 1 \tag{1.10}$$

Although the magnetohydrodynamic forces and Joule heat are zero at $x < 0$ (because of the above simplification of the flow), the gasdynamic parameters $(p, \rho, \mathbf{v})$ can generally vary in this region because the acoustic perturbations travel upstream. The cross section to the left of the cross section $x = 0$ is found to be unperturbed (by the magnetic field) and homogeneous if the absolute velocity of the gas is supersonic $(M_0 > 1)$ behind the shock wave until it enters zone L. The parameters corresponding to this case, namely $\rho = \rho_0$, $p = p_0$ and $u = u_0$ can be found from (1.6) by setting $D = 1$ and $v_n = u_0$.

On the other hand, if the absolute velocity of the gas behind the shock wave is lower than the velocity of sound $(M_0 < 1)$. then the gas parameters to the left of the cross section $x = 0$ are not equal to ρ_0, p_0 and u_0, and can be found only from the general solution of our problem.

The unperturbed parameters ρ_0, p_0 and u_0, corresponding to the gas motion behind a rectilinear shock wave travelling at a velocity $D_* = \text{const}$ are defined by the following formulas:

$$\rho_0 = \left(1 + \frac{2s}{\gamma - 1}\right)^{-1}, \qquad u_0 = \frac{2(1-s)}{\gamma+1}$$
$$p_0 = \frac{(\gamma-1)s}{\gamma(\gamma+1)} + \frac{2(\gamma-1)(1-s)}{(\gamma+1)^2}, \qquad v_0 = 0 \tag{1.11}$$
$$a_0^2 = \gamma p_0/\rho_0, \ M_0 = u_0/a_0$$

In the upstream end zone of the magnetic field it is also possible to find a cross section $x = x^*$, to the right of which $\mathbf{j}$ is always equal to zero and φ is always constant. Hence, at the instants when the shock wave is to the right of the zone $0 < x < x^*$, we must consider Eqs. (1.5) for this zone with due allowance for (1.10) and for the following boundary condition:

$$j_x = 0 \quad \text{at} \quad x = x^*, 0 < y < 1 \tag{1.12}$$

On the other hand, when the shock wave travels through the zone $0 < x < x^*$, our consideration of Eqs. (1.5) in the region between $x = 0$ and the shock wave itself must take into account boundary conditions (1.7), (1.8) and (1.10).

The set of equations (1.1) – (1.4) serves to define the gasdynamic parameters behind the shock wave. Relations (1.6), (1.8) and (1.11) are the boundary conditions for the solution of set (1.1) – (1.4), and for finding the velocity and shape of the shock wave.

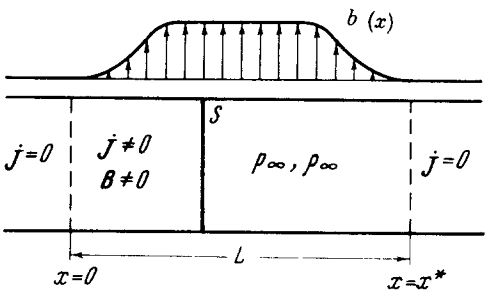


Fig. 2

The fundamental complexity of our problem is due to the need for simultaneous solution of "electrical" equations (1.5) and "gasdynamic" equations (1.1) – (1.4). It should be also mentioned that Eqs. (1.5) are elliptical, while (1.1) – (1.4) are hyperbolic. This often complicates the application of methods of numerical analysis still further.

However, if the parameter of dynamic interaction between the gas and magnetic field is fairly small, then Eqs. (1.5) and (1.1) – (1.4) can be solved successively. In this case, we first find the distribution of electrical currents from (1.5), assuming that the gasdynamic parameters are defined by (1.11), and that the shock wave is rectilinear and travels at the velocity $D = 1$. In the second stage we analyze Eqs. (1.1) – (1.4) assuming that the quantities $\mathbf{j} \times \mathbf{b}$ and $\mathbf{j}^2 \sigma$ which appear in these equations are known, having been calculated from the earlier solution of (1.5). This method will be used below.

In carrying out actual computations we must remember that the cross sections $x = 0$ and $x = x^*$ must be chosen in accordance with the specific features of the magnetic field distribution. For example, if $\mathbf{b} = (0, 0, b(x))$ and $b(x)$ are approximated by a step-function (Fig. 1), then the length x^* must be greater than L. On the other hand, if $b(x)$ asymptotically tends to 0 for $|x| \to \infty$, it is always possible to choose L in such a way that $L = x^*$ (Fig. 2).

2. When the interaction parameter N between the dynamic gas and the magnetic field is small, the solution of the set of equations (1.1) – (1.8) can be sought in the form of the following series;

$$u = u_0 + Nu_1(x, y, t) + \ldots, \qquad v = Nv_1(x, y, t) + \ldots$$
$$\rho = \rho_0 + N\rho_1(x, y, t) + \ldots, \qquad p = p_0 + Np_1(x, y, t) + \ldots \qquad (2.1)$$
$$\mathbf{j} = \mathbf{j}_0(x, y, t) + N\mathbf{j}_1(x, y, t) + \ldots, \qquad \varphi = \varphi_0(x, y, t) + $$
$$+ N\varphi_1(x, y, t) + \ldots$$
$$\Phi(x, y, t) = x - t + N\psi(y, t) + \ldots = 0$$

Here the constants u_0, p_0 and ρ_0 are defined by formulas (1.11). Equation $\Phi = 0$ defines the shape of the shock wave.

Substituting (2.1) into (1.1) – (1.8) we obtain the following equations defining the electromagnetic parameters $\mathbf{j}_0$ and φ_0 and the gasdynamic parameters u_1, v_1, p_1 and ρ_1:

$$\mathbf{j}_0 = \sigma_0(-\nabla\varphi_0 + \mathbf{v}_0 \times \mathbf{b}), \qquad \operatorname{div} j_0 = 0 \qquad (2.2)$$
$$j_{x0} = 0 \quad \text{at} \quad x = 0$$

$$j_{n0} = 0 \quad \text{at the shock wave} \quad (\text{or } j_{x0} = 0 \quad \text{at} \quad x = x^*)$$
$$j_{y0} = 0 \quad \text{at} \quad y = 0, \quad y = 1$$
$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial \rho_1}{\partial x} + \rho_0 \frac{\partial v_1}{\partial y} + \frac{\nu \rho_0 v_1}{y} = 0$$
$$\rho_0 \frac{\partial u_1}{\partial t} + \rho_0 u_0 \frac{\partial u_1}{\partial x} + \frac{\partial p_1}{\partial x} = f_x \qquad (f_x = (\mathbf{j}_0 \times \mathbf{b})_x) \qquad (2.3)$$
$$\rho_0 \frac{\partial v_1}{\partial t} + \rho_0 u_0 \frac{\partial v_1}{\partial x} + \frac{\partial p_1}{\partial y} = f_y \qquad (f_y = (\mathbf{j}_0 \times \mathbf{b})_y)$$
$$\frac{\partial p_1}{\partial t} + u_0 \frac{\partial p_1}{\partial x} + \gamma p_0 \left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\nu v_1}{y}\right) = (\gamma - 1) q \qquad \left(q = \frac{\mathbf{j}_0^2}{\sigma}\right)$$
$$p_1 = \alpha u_1, \quad \rho_1 = \beta u_1, \quad \frac{\partial \psi}{\partial t} = -\delta u_1, \quad v_1 = u_0 \psi(y, t) \qquad (2.4)$$
$$\alpha = \frac{2(\gamma - 1)}{(\gamma + 1)(1 + s)}, \quad \beta = \frac{2s(\gamma^2 - 1)}{(\gamma - 1 + 2s)^2(1 + s)}, \quad \delta = \frac{\gamma + 1}{2(1 + s)}$$
$$v_1 = 0 \quad \text{at} \quad y = 0, \ y = 1 \qquad (2.5)$$

Let us average Eqs. (2.3) – (2.5) with respect to y. Allowing for (2.5), we obtain the following set of equations for the gasdynamic parameters averaged over the cross section:

$$\frac{\partial \langle \rho_1 \rangle}{\partial t} + \rho_0 \frac{\partial \langle u_1 \rangle}{\partial x} + u_0 \frac{\partial \langle \rho_1 \rangle}{\partial x} = 0$$
$$\rho_0 \frac{\partial \langle u_1 \rangle}{\partial t} + \rho_0 u_0 \frac{\partial \langle u_1 \rangle}{\partial x} + \frac{\partial \langle p_1 \rangle}{\partial x} = \langle f_x \rangle$$
$$\frac{\partial \langle p_1 \rangle}{\partial t} + u_0 \frac{\partial \langle p_1 \rangle}{\partial x} + \gamma p_0 \frac{\partial \langle u_1 \rangle}{\partial x} = (\gamma - 1) \langle q \rangle \qquad (2.6)$$
$$\langle p_1 \rangle = \alpha \langle u_1 \rangle, \quad \langle \rho_1 \rangle = \beta \langle u_1 \rangle, \quad -\left\langle \frac{\partial \psi}{\partial t} \right\rangle = \delta \langle u_1 \rangle \quad \text{at} \quad x = t \quad (2.7)$$
$$\left(\langle \xi \rangle = \int_0^1 \xi(x, y, t)\, y^\nu\, dy\right)$$

All of the averaged parameters above are functions of x and t; the quantity $-\langle \partial\psi / \partial t \rangle$ characterizes the perturbation of the average velocity of the shock wave

$$\langle D\rangle = 1 - N\,\langle \partial\psi / \partial t\rangle + \ldots \tag{2.8}$$

The method of characteristics which has been so successfully used for the solution of many unsteady one-dimensional problems in gasdynamics and magnetogasdynamics also constitutes the simplest way of analyzing Eqs. (2.6) in this case.

Equations (2.6) have the three following real families of characteristics:

$$\frac{dx}{dt} = u_0 \pm a_0 \qquad \left(a_0{}^2 = \frac{\gamma p_0}{\rho_0}\right)$$

$$\pm\, d\,\langle p_1\rangle + \rho_0 a_0 d\,\langle u_1\rangle - dt\,[a_0\,\langle f_x\rangle \pm (\gamma - 1)\,\langle q\rangle] = 0 \tag{2.9}$$

$$\frac{dx}{dt} = u_0, \qquad d\,\langle \rho_1\rangle = -\,\rho_0\,\frac{\partial\,\langle u_1\rangle}{\partial x}\,dt \tag{2.10}$$

Let us point out that our gasdynamic problem in this case can be solved in successive stages by means of characteristic relations (2.9), (2.10). First, the quantities $\langle u_1\rangle$, $\langle p_1\rangle$ and the perturbed shock wave velocity must be found from (2.9) under suitable boundary conditions; then the gas density $\langle \rho_1\rangle$ must be determined from (2.10).

We assume that the magnetic field decays asymptotically, so that the system shown in Fig. 2 is valid. The corresponding families of characteristics (2.9), (2.10) for $M_0 > 1$ are shown in Fig. 3. Straight lines 1 and 2 are the characteristics of the first and second families; straight lines 3 are the trajectories of the gas particles.

First, we find the gas parameters immediately behind the shock wave (for instance, at point A, Fig. 3). Allowing for the relations (2.9) which connect the parameters along characteristic $A'A$, for the boundary conditions (2.7) and for the following equations:

$$\langle u_1\rangle = 0, \quad \langle p_1\rangle = 0, \quad \langle \rho_1\rangle = 0 \quad \text{at} \quad x = 0$$

we obtain

$$\langle u_{1A}\rangle = (u_0 + a_0)\,(\alpha + \rho_0 a_0) \int\limits_{A'A} G\,dx \qquad (G = a_0\,\langle f_x\rangle + (\gamma - 1)\,\langle q\rangle)$$

$$\langle p_{1A}\rangle = \alpha\,\langle u_{1A}\rangle, \qquad \langle \rho_{1A}\rangle = \beta\,\langle u_{1A}\rangle, \qquad -\left\langle\frac{\partial\psi}{\partial t}\right\rangle = \delta\,\langle u_{1A}\rangle \tag{2.11}$$

The gas parameters at an arbitrary point of flow C can now be found by means of (2.9), (2.10) and (2.11).

After some elementary transformations we obtain

$$\langle p_{1C}\rangle = \frac{(\alpha - \rho_0 u_0)}{2\,(u_0 + a_0)\,(\alpha + \rho_0 a_0)} \int\limits_{A'A} G\,dx + \frac{1}{2\,(u_0 + a_0)} \int\limits_{K'C} G\,dx - \frac{1}{2\,(u_0 - a_0)} \int\limits_{AC} G^*\,dx$$

$$\langle u_{1C}\rangle = \frac{1}{\rho_0 a_0}\left(\frac{1}{u_0 + a_0} \int\limits_{K'C} G\,dx - \langle p_{1C}\rangle\right) \tag{2.12}$$

$$\langle \rho_{1C}\rangle = \langle \rho_{1R}\rangle - \frac{\rho_0}{u_0} \int\limits_{RC} \frac{\partial\,\langle u_1\rangle}{\partial x}\,dx \qquad (G^* = a_0\,\langle f_x\rangle - (\gamma - 1)\,\langle q\rangle)$$

Quantity $\langle \rho_{1R}\rangle$ equals to zero if the trajectory of particles passing through point C intersects axis t; it is equal to the perturbation of gas density at point R' in the shock wave if the trajectory of particles passing through the considered point C' intersects the shock wave (Fig. 3).

The gasdynamic parameters for $M_0 < 1$ can be found in the same way. In this case the region of perturbed gas motion in the plane x, t lies between the straight lines $t = x$

and $t = x/(u_0 - a_0)$. On the second of these two lines, the perturbation of gasdynamic parameters is equal to zero. The parameters of the gas directly behind the shock wave are defined by (2.11), as in the earlier case $M_0 > 1$, since the perturbing factors are equal to zero when $x < 0$, and integration along the characteristic must be carried out over the interval $0 < x < x^*$.

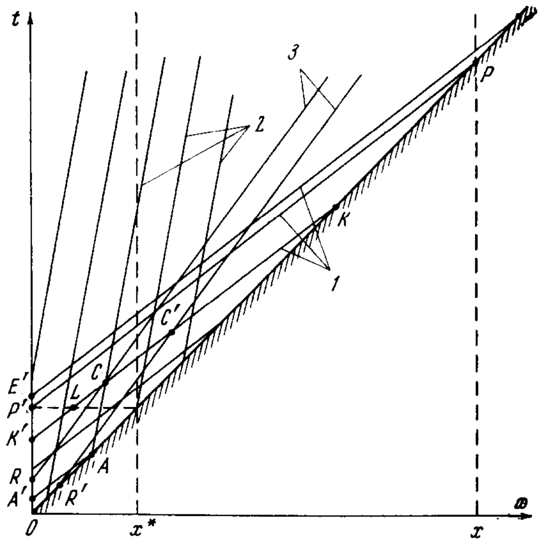


Fig. 3

Thus, the averaged gasdynamic parameters behind the shock wave can be determined relatively easily when we know the perturbing factors, i.e. the magnetohydrodynamic force $\langle f_x \rangle$ and the Joule heat $\langle q \rangle$. These factors have to be found by solving Eqs. (2.2).

3. First of all, we shall find the propagation velocity of the shock wave a large distance downstream from the zone of the magnetic field.

Let us consider the position of the shock wave defined by the following conditions:

$$x = \zeta > x^{**}, \qquad x^{**} = x^* \frac{u_0 + a_0}{u_0 + a_0 - 1} \tag{3.1}$$

It is clear (Fig. 3) that quantities $\langle f_x \rangle$ and $\langle q \rangle$ on the corresponding characteristic $E'E$ do not depend on t and are functions of x alone. This is due to the electric field in the channel becoming stationary when the shock wave leaves the region $0 < x < x^*$.

Integration along characteristic $E'E$ yields

$$\int_{E'E} G\, dx = a_0 \int_0^{x^*} \langle f_x \rangle\, dx + (\gamma - 1) \int_0^{x^*} \langle q \rangle\, dx \tag{3.2}$$

It should be pointed out that quantities $\langle f_x \rangle$ and $\langle q \rangle$ in (3.2) represent the perturbing factors corresponding to a steady gas flow (with velocity u_0 and conductivity σ_0) through the channel in the specified nonhomogeneous magnetic field. When a conducting medium flows through a channel with electrically nonconducting walls, the following integral equation holds [3]:

$$-u_0 \int_0^{x^*} \langle f_x \rangle\, dx = \int_0^{x^*} \langle q \rangle\, dx = q_\Sigma \tag{3.3}$$

Here q_Σ is the total dimensionless Joule dissipation in the channel.

Using (2.11), (3.2), (3.3) and (2.8) we obtain the following asymptotic expression for the propagation velocity of the shock wave:

$$\langle D^\circ \rangle - D_* = -\frac{\sigma_0^\circ B_*^2 h}{c^2 \rho_\infty} T(s, \gamma)\, Q_\Sigma, \qquad T(s, \gamma) = \frac{(\gamma - 1)\, \delta u_0 \left[a_0 - (\gamma - 1) u_0\right]}{(\gamma + 1)(u_0 + a_0)(\alpha + \rho_0 a_0)}$$

$$q_\Sigma = \sigma_0 u_0^2 Q_\Sigma \tag{3.4}$$

Here $\langle D^\circ \rangle$ is the averaged dimensional velocity of the shock wave, σ_0° the dimensional conductivity calculated from parameters (1.11), and Q_Σ the dimensionless Joule dissipation corresponding to a gas flow for which both velocity and conductivity are equal unity. The quantities δ, u_0, a_0, ρ_0 and α are obtainable from (1.11) and (2.4).

The Joule dissipation Q_Σ is one of the most important characteristics of the magnetohydrodynamic systems; it has been calculated by several authors. Formula (3.4) indicates that Q_Σ enables us to find the resulting change in the velocity of the shock wave.

Let us now investigate some properties of function $T(s, \gamma)$. First of all, we notice that parameter s varies between 0 (there is no counter-pressure, $M_\infty = (D_* / a_\infty) \to \infty$) and unity (a weak discontinuity spreads through the gas, $M_\infty = 1$, $M_0 = 0$).

We shall prove that the function $T(s, \gamma) > 0$ for $1 < \gamma < 2$. For this purpose, we first show that $r(s, \gamma) = 1 - (\gamma - 1)M_0 > 0$. This inequality is easy to prove by noting that $\partial r / \partial s = -(\gamma - 1)\partial M_0 / \partial s > 0$ for $1 < \gamma < 2$, so that only fulfilment of condition $r(0, \gamma) > 0$ needs to be verified. Using (1.11) it is easy to show that $r(0, \gamma)$ is always larger than 0, provided the value of γ remains within the specified range of variation.

Thus, when inequality $1 < \gamma < 2$ is valid, interaction between the shock wave and magnetic field must result in a reduction of the propagation velocity of the wave.

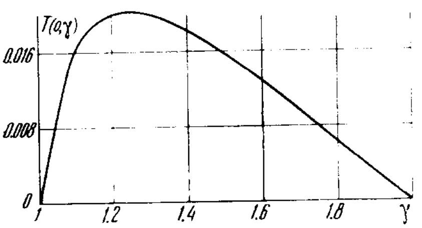


Fig. 4

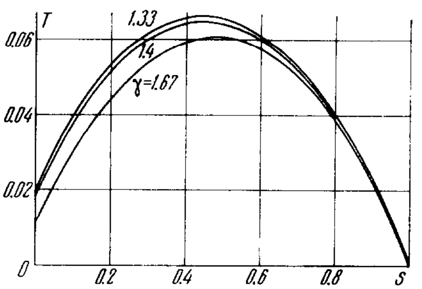


Fig. 5

To illustrate the effect of the parameters γ and s on the asymptotic velocity of the shock wave, we have in Figs. 4 and 5 the values of $T(0, \gamma)$ and $T(s, \gamma)$ for various values of γ. We note that $T(1, \gamma) = 0$, $T(s, 1) = 0$, $T(0, 2) = 0$, and, when the value of γ is constant, there is a value of $s = s_{\max}(\gamma)$ such that the shock wave experiences the maximum retardation.

Shock waves of infinitely high intensity ($s = 0$) are of the greatest practical interest. All available information leads to the conclusion that the propagation of such shock waves resembles most closely the motion of plasmoids in electrical discharge devices. The function $T(0, \gamma)$ shown in Fig. 4 enables us to elucidate the effect of magnitude of γ on the velocity of shock waves in this case.

Let us now consider several examples of shock wave propagation in nonhomogeneous magnetic fields and determine the velocity of the shock front for arbitrary values of its coordinate ζ.

4. We consider a shock wave travelling in a circular tube ($\nu = 1$) across an externally applied axisymmetric magnetic field (the cylindrical coordinate system y, θ, x is used)

$$\mathbf{b} = (b_y(x, y), 0, b_x(x, y)) \tag{4.1}$$

In this case $\nabla\varphi \equiv 0$, and the distribution of electromagnetic parameters in accordance with (2.2) is represented by the following formulas:

$$\mathbf{j} = (0, -\sigma_0 u_0 b_y, 0), \qquad f_x = -\sigma_0 u_0 b_y^2, \qquad \langle f_x \rangle = -\sigma_0 u_0 g(x)$$
$$q = \sigma_0 u_0^2 b_y^2, \quad \langle q \rangle = \sigma_0 u_0^2 g(x) \quad \left(g(x) = \int_0^1 y b_y^2 dy\right) \tag{4.2}$$

Relations (4.1) are valid for the region between the cross section $x = 0$ and the shock wave.

A particularly interesting characteristic of the distribution of electromagnetic parameters is that the current density at all points of space through which the shock wave has already passed is independent of time.

From (2.11) and (3.2) we obtain the following expression for the velocity of the shock wave:

$$\langle D^\circ \rangle - D_* = -\frac{\sigma_0^\circ B_*^2 h}{c^2 \rho_\infty} T(s, \gamma) K(\zeta), \qquad K(\zeta) = \int_0^\zeta g(x)\, dx \tag{4.3}$$

Quantities D°, σ_0°, and T in (4.3) can be found in the same way as in formulas (3.4); $x = \zeta$ is the present coordinate of the shock wave.

It follows from (4.3) and inequalities $g \geqslant 0$ and $T \geqslant 0$ that the propagation of a shock wave in a circular tube in the presence of an axisymmetric magnetic field is accompanied by continuous retardation of the shock front.

For the convenience of numerical computation, the coordinate of some characteristic point in the region of the magnetic field will be designated x_0; we also introduce the new variable $x - x_0 = X$. Allowing for the asymptotic nature of the magnetic field decay, we assume that $x_0 \gg 1$. We then have

$$K = \int_{-x_0}^{\zeta - x_0} g\,dX \approx \int_{-\infty}^{\zeta - x_0} g\,dX \tag{4.4}$$

As an example, let us consider the propagation of a shock wave in a magnetic field which is nearly axisymmetric within the channel. This is the case if it may be assumed that the tube radius h is considerably smaller than the characteristic radius H of the solenoidal winding ($\varepsilon = h / H \ll 1$).

In this case, the following expression may serve as an approximation of b_y [6]:

$$b_y = -\frac{\tau'(\xi)\,\eta}{2} + O(\eta^3) \qquad \left(\tau' = \frac{d\tau}{d\xi}, \quad \xi = \frac{X^\circ}{H}, \quad \eta = \frac{y^\circ}{H}\right) \tag{4.5}$$

where $\tau(\xi)$ is the distribution of the axial field component b_x for $y = 0$.

Using (4.5), we can write the integral in (4.4) as follows:

$$K = \int_{-\infty}^{\zeta - x_0} g\,dX = \frac{\varepsilon}{16} \int_{-\infty}^{\mu} \tau'^2(\xi)\, d\xi + O(\varepsilon^3) \qquad \left(\varepsilon = \frac{h}{H} \ll 1, \mu = \frac{\zeta^\circ - x_0^\circ}{H}\right) \tag{4.6}$$

Let us carry out the computations for a magnetic field generated by a single current-carrying turn of radius H, placed at the cross section $\xi = 0$, and also for a field whose origin is a semi-infinite solenoidal winding of the same radius. For these two fields, function $\tau(\xi)$ is expressed by the formulas

$$\tau(\xi) = (1+\xi^2)^{-3/2} \quad \text{(single turn)}, \quad \tau(\xi) = {}^1/_2\,(1+\xi(1+\xi^2)^{-1/2}) \quad \text{(solenoid)}$$

The field at the point $\xi = 0$, $\eta = 0$ is taken as the characteristic field B_* generated by a single turn, and the magnetic induction for $\xi = \infty$ is chosen to represent the characteristic field of the solenoidal winding.

Quantity K in these two cases is

$$K = K(\mu) = \frac{9\varepsilon}{16\cdot 8\cdot 48}\left[15\left(\operatorname{arctg}\mu + \frac{\pi}{2}\right) + \frac{15\mu^7 + 55\mu^5 + 73\mu^3 - 15\mu}{(1+\mu^2)^4}\right] + O(\varepsilon^3) \quad \text{(single turn)}$$

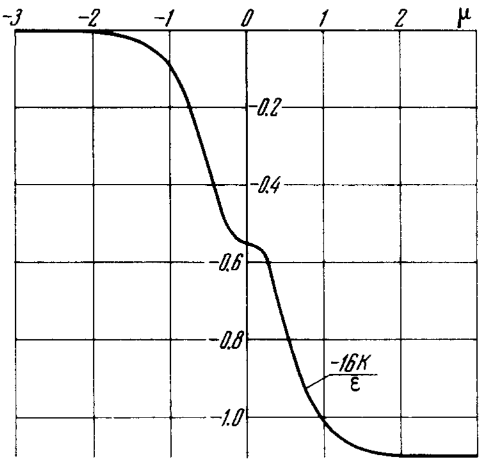
$$K = K(\mu) = \frac{\varepsilon}{16\cdot 4}\left[\frac{5\mu + 3\mu^3}{8(1+\mu^2)^2} + \frac{3}{8}\left(\frac{\pi}{2} + \operatorname{arctg}\mu\right)\right] + O(\varepsilon^3) \quad \text{(solenoid)}$$

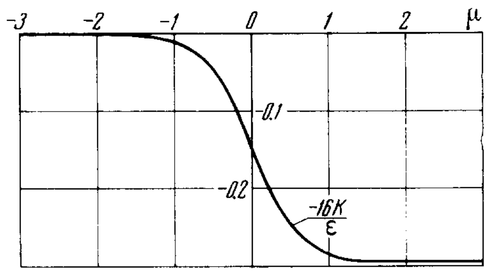
Figure 6 shows the ratios $16K/\varepsilon$ illustrating the retardation of shock wave S in the magnetic fields of a single turn and of a solenoidal winding (the terms $O(\varepsilon^3)$ are neglected).

We note that $K_{max} = K(\infty) = 9 \times 15\,\varepsilon\pi / (16 \times 8 \times 48)$ in the case of the magnetic field induced by a single turn, and that $K_{max} = K(\infty) = 3\pi\varepsilon / (16 \times 32)$ when the shock wave travels through the field generated by a solenoidal winding.

The graphs in Fig. 6 show that the shock wave is retarded equally in the regions $\xi < 0$ and $\xi > 0$.

5. Now let us turn to the propagation of a shock wave in a flat channel





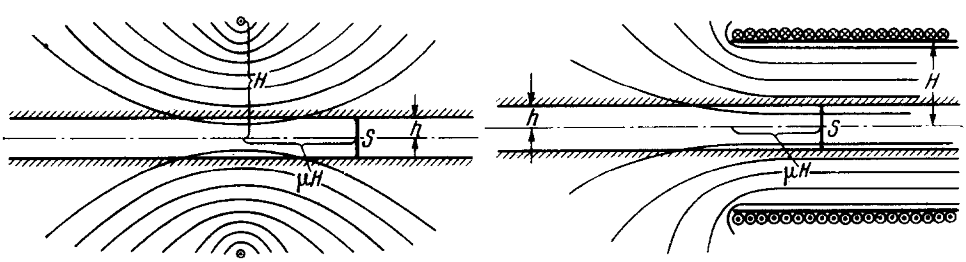


Fig. 6

($\nu = 0$) in the presence of a magnetic field $\mathbf{b}=(0,0,b(x))$. We assume that the computation as in Fig. 2 may be used. In contrast to the case of flow in an axisymmetric magnetic field, in the present case the distribution of electromagnetic parameters of the flow is much more complex in the regions $0 < x < t$ (for $t < x^*$) and $0 < x < x^*$ (for $t > x^*$). In this case we must solve the following boundary value problem in order to determine the current j:

$$\Delta\omega = 0, \quad \partial\omega/\partial y = -b(x) \quad \text{at} \quad y = 0,\ y = 1 \tag{5.1}$$

$$\frac{\partial\omega}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad \begin{cases} x = t & (\text{when } t < x^*) \\ x = x^* & (\text{when } t > x^*) \end{cases}$$

$$\left(\mathbf{j}_0 = \sigma_c u_0 \mathbf{J}, \quad \varphi_0 = u_0\omega, \quad J_x = -\frac{\partial\omega}{\partial x}, \quad J_y = -\frac{\partial\omega}{\partial y} - b(x)\right)$$

Attention should be drawn to the fact that the time t characterizes the position of the shock wave (*) and it is a parameter in equations (5.1). If $t < x^*$, problem (5.1) has to be solved for the region $0 < x < t$, $0 < y < 1$, and quantity $\mathbf{J}$ turns out to be dependent on t: $\mathbf{J} = \mathbf{J}(x, t)$. However, if $t > x^*$, then Eq. (5.1) must be considered in the region $0 < x < x^*$, $0 < y < 1$, so $\mathbf{J} = \mathbf{J}(x)$.

The quantities $\langle f_x\rangle$ and $\langle q\rangle$ and the integral in (2.11) can be calculated by means of relations (2.3) as follows:

$$\langle f_x\rangle = \sigma_0 u_0 F, \qquad F = F(x,t) = \int_0^1 J_y b\, dy \tag{5.2}$$

$$\langle q\rangle = \sigma_0 u_0^2 Q, \qquad Q = Q(x,t) = \int_0^1 (J_x^2 + J_y^2)\, dy, \qquad \int_{A'A} G\, dx = \sigma_0 u_0 R(\zeta), \qquad \zeta = x(A)$$

$$R = R(\zeta) = \int_0^\zeta [a_0 F(x,z) + (\gamma - 1) u_0 Q(x,z)]\, dx, \qquad z = \zeta - \frac{\zeta - x}{u_0 + a_0}$$

The perturbations of the gasdynamic parameters in the shock wave are found from (2.11), with allowance for (5.2) and relations (1.11) and (2.4) which serve to define u_0, a_0, α, β and δ. The velocity of the shock wave can be expressed as follows:

$$\langle D^\circ\rangle - D_* = \frac{\sigma_0{}^\circ B_*{}^2 h}{c^2\rho_\infty} \frac{T(s,\gamma)}{[a_0 - (\gamma - 1) u_0]} R(\zeta) \qquad (0<\zeta<\infty)$$

$$D^\circ = D_* \qquad (\zeta<0) \tag{5.3}$$

When the function R is calculated from (5.2), it is necessary to distinguish between the following intervals in which ζ varies: $(0, x^*)$, (x^*, x^{**}) and $x > x^{**}$, where x^{**} is defined by formula (3.1). In the first of these intervals, the functions F and Q on a specified characteristic $A'A$ depend on two arguments, namely x and t. When ζ lies in the second interval (e. g. the point K in Fig. 3), the characteristic $K'K$ includes a point L in which it intersects the straight line $t = x^*$, and this point divides $K'K$ into the section $K'L$ where $F = F(x, t)$ and $Q = Q(x, t)$, and the section LK where $F = F(x)$ and $Q = Q(x)$ for $x(L) < x < x^*$, and $F = 0$, $Q = 0$ for $x^* < x < x(K)$.

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*) It must be remembered that equation $x(t) = t$ represents the law of motion of the shock wave for $N = 0$.

However, we already showed in Sect. 3 that for $x > x^{**}$ the functions F and Q on the integration path along the characteristic depend solely on x from $x = 0$ to $x = x^*$, and are equal to zero for $x > x^*$. In this case

$$R(\zeta) = -[a_0 - (\gamma - 1) u_0] Q_\Sigma, \qquad Q_\Sigma = \int_0^{x^*} Q(x)\, dx$$

and (5.3) becomes (3.4).

For example, we investigated the propagation of the shock wave in a magnetic field, defined as follows: (5.4)

$$b(x) = \begin{cases} \exp[-(x-3)^2] & (0<x<3) \\ 1 & (3<x<4) \\ \exp[-(x-4)^2] & (4<x<7) \end{cases}$$

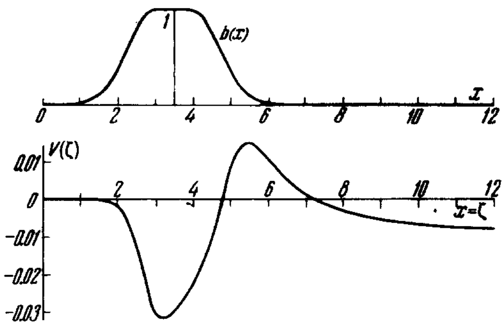

Fig. 7

This magnetic field is illustrated in Fig. 7.

Preliminary computations of steady electric fields for a specified geometry of the magnetic field indicated that electrical currents were virtually zero at the cross sections $x = 0$ and $x = 7$. For the purposes of finding the solution of problem (5.1) we therefore assumed that $x^* = 7$.

Equations (5.1) were solved by numerical analysis for an arbitrary position of the front boundary of that region in which the electrical currents flow. From the values of $F(x, t)$ and $Q(x, t)$, determined in this manner, we found the perturbed gas parameters behind the shock wave, and also calculated the velocity of the shock front.

The dynamic conditions of shock wave propagation in a channel in the presence of a magnetic field are illustrated in Fig. 7, which shows the function $V(\zeta)$, calculated for $\gamma = 1.42$ and $s = 0$ from the following expression (see formula (5.3)):

$$\langle D^\circ \rangle - D_* = 0.204 \frac{\sigma_0^\circ B_*^2 h}{c^2 \rho_\infty} V(\zeta) \qquad \text{for } 0 < \zeta < \infty$$
$$D^\circ = D_* \qquad \text{for } \zeta < 0 \tag{5.5}$$

Figure 7 indicates that the velocity of the shock wave changes extremely nonmonotonically along the channel. When the shock front moves through the upstream end zone of magnetic field ($\zeta < 3.5$), a reduction in its velocity occurs ($V < 0$, $dV / d\zeta < 0$). This effect is connected with the formation of terminal current in the region $0 < x < 3.5$; this current is oriented in such a way that the force exerted on gas layers near shock wave S retards them (Fig. 8a).

Further, when the shock wave enters the upstream end zone of magnetic field ($\zeta > 3.5$), its velocity begins to increase ($dV / d\zeta > 0$). This type of dependence may be explained by the emergence of a second turn of terminal current which interacts with the magnetic field in such a manner that the force exerted on the gas layers near the shock wave S accelerates them (Fig. 8b). The corresponding perturbations catch up with the shock wave and accelerate it. For some value of ζ the quantity V reaches its maximum value $V_{max} > 0$ and then begins to drop asymptotically down to $V = V_\infty = -T(0, 1.42) Q_\Sigma / 0.204$. This reduction in the shock wave velocity can be explained by the perturbing factors $\langle f_x \rangle$ and $\langle q \rangle$ being virtually equal to zero directly behind the front of shock wave occupying positions corresponding to $\zeta > 5$, while the total effect of perturbations generated within region $0 < x < x^*$ is to retard the shock wave (as has been shown in Sect. 3).

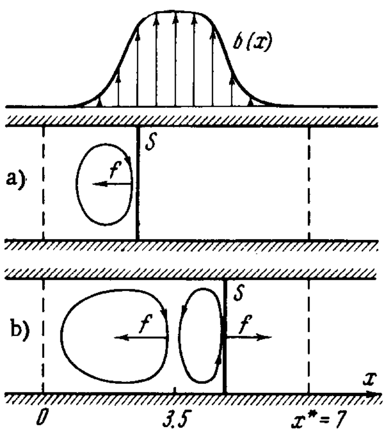

Fig. 8

The acceleration of the front of travelling plasma in the exit zone of the magnetic field (*) was noted in experiments [4, 5].

Computations for $\gamma = 1.67$ and $s = 0$ showed that the nature of the shock wave motion remains the same.

6. We have already noted that expressions (2.11) and (2.12) enable us to calculate the perturbation of gasdynamic parameters, once the forces and heat sources, $\langle f_x \rangle$ and $\langle q \rangle$, have been found. The distribution of these quantities becomes known when we solve the equations for electrical parameters $(\mathbf{j}_0, \varphi_0)$, expressed in their simplest form in (2.2). Sets of equations such as (2.2) have been widely investigated for a long time (see [3]). There is, therefore, a good chance of making use of already established results and computational methods in studying shock wave motion in channels with anisotropic conductivity, induced magnetic fields, generating and accelerating flow modes, etc.

From the available information concerning the effects of various parameters on integral characteristics and magnetohydrodynamic devices, a number of conclusions can be drawn about the amount of retardation experienced by a shock wave in a magnetic field. First, it must be pointed out that the Joule dissipation Q_{Σ} in a gas flow in a channel with nonconducting walls decreases as the Hall parameter β increases and the Reynolds magnetic number R_m decreases. Hence, in accordance with (3.4), as β and R_m increase, the dimensionless characteristic of retardation of the shock wave $\epsilon(\infty)$, where $\epsilon(\zeta) = (\langle D^{\circ} \rangle - D_*) / D_* N$, also decreases. The corresponding curves $\epsilon = \epsilon(\zeta)$ must steadily shift to the right as R_m increases; this is a direct consequence of the magnetic force

*) In further experiments (reported to me orally) the author of [4, 5] noticed that the velocity of the plasma front behind the zone of a nonhomogeneous magnetic field of moderate intensity (for $N < 1$) rises in excess beyond its original value D_*, and then drops again below D_*. It is evident that the same effect is revealed in our theoretical analysis of the unsteady gas motion. The numerical discrepancy between our analytical results and the experiments can be explained, above all, by the induced magnetic fields (present in the experiments) and the use (in our analysis) of relations at the shock wave as boundary conditions on the plasma front.

Satisfactory qualitative agreement between the analytical and experimental results suggests that many properties of magnetic gasdynamic flows in pusle discharge devices and shock tubes might be explained by conventional ideas of gasdynamics.

lines being swept along by the gas stream.

On considering the motion of shock wave in a channel of a magnetohydrodynamic generator, we must remember that the kinetic energy of the gas is converted not only into electrical energy P applied to the external load, but also into the Joule heat Q. Using the general energy equation $A = Q + P$ which relates the work A done by the medium in overcoming the counterforce of magnetic field, to the quantities Q and P, we can prove that the asymptotic expression for the velocity of shock wave is of the form

$$\langle D^\circ\rangle - D_* = -\frac{\sigma_0{}^\circ B_*{}^2 h}{c^2\rho_\infty}\frac{T(s,\gamma)}{[a_0-(\gamma-1)u_0]}\{Q_\Sigma[a_0-(\gamma-1)u_0]+a_0P_\Sigma\} \qquad (6.1)$$

$$P = \frac{\sigma_0{}^\circ B_*{}^2 h^2 D_*{}^2}{c^2}u_0{}^2 P_\Sigma$$

The quantities occurring in (6.1) are defined in (3.4).

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